

A NUMERICAL APPROACH TO SOME BASIC THEOREMS IN SINGULARITY THEORY

TA LÊ LOI AND PHAN PHIEN

ABSTRACT. In this paper, we give the explicit bounds for the data of objects involved in some basic theorems of Singularity theory: the Inverse, Implicit and Rank Theorems for Lipschitz mappings, Splitting Lemma and Morse Lemma, the density and openness of Morse functions. We expect that the results will make Singularities more applicable and will be useful for Numerical Analysis and some fields of computing.

1. INTRODUCTION

To make Singularity Theory more applicable it is important to make its basic results ‘quantitative’. This direction of the theory is proposed by Y. Yomdin in [Y1] where he proves the quantitative Morse-Sard theorem by giving the notion of near-critical values of differentiable mappings and estimated these sets by the metric entropy. For the discussion on this direction and its developments we refer the readers to [Y2] and [Y-C] and the references therein. We are interested in the numerical approach of this direction. In this paper, we give the quantitative versions of some basic theorems of Singularity theory: the Inverse, Implicit and Rank Theorems for Lipschitz mappings, Splitting Lemma and Morse Lemma, the density and openness of Morse functions. The explicit bounds for the data of the objects involved are estimated via the input data (e.g. C^k -norms, radii of the balls, ...). The main tools that we use are some familiar methods of Singularities of differentiable mappings (see [A-G-V], [B-L], [G-G], or [Ma]), the quantitative forms of the Inverse and Implicit mappings theorems, and Morse-Sard theorem (see [C1], [Pa], [P], Theorems 3.1 and 3.5 in this paper, [Y1] and [Y-C]). In our results, the estimates of the first order derivatives and the radii of the domains of the mappings involved are quite sharp. Since we use Lemmas 2.3 and 2.4, the estimates of the higher order derivatives of the mappings involved are explicit but rather big. We expect that the results will make Singularities more applicable and will be useful for Numerical Analysis and some fields of computing.

The plan of our paper is as follows: In Section 2 we recall some definitions and give the estimates of C^k -norms of compositions and inverses. In Section 3 we consider

2010 *Mathematics Subject Classification.* Primary 14B05; Secondary 65D25, 34A55.

Key words and phrases. Lipchitz mappings, Rank Theorem, Splitting Lemma, Morse functions.

the quantitative versions of the Inverse, Implicit and Rank Theorems for Lipschitz mappings. In Section 4 we give a quantitative form of diagonalization of matrix-valued mappings by upper triangular matrices, the quantitative versions of Splitting Lemma and Morse Lemma, and applications to the density and openness of Morse functions on a ball.

2. PRELIMINARIES

We give here some definitions, notations and results that will be used later.

Let $\mathbf{M}_{m \times n}$ denote the vector space of real $m \times n$ matrices,

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}, \text{ where } x \in \mathbb{R}^n,$$

$\mathbf{B}_r^n(x_0)$ denotes the ball of radius r , centered at x_0 in \mathbb{R}^n , $\mathbf{B}_r^n = \mathbf{B}_r^n(0)$, and $\mathbf{B}^n = \mathbf{B}_1^n$,

$$\|A\| = \max_{\|x\|=1} \|Ax\|, \text{ where } A \in \mathbf{M}_{m \times n}, \text{ or } A \text{ is a linear mapping.}$$

$\mathcal{B}_{m \times n}$ denotes the unit ball in $\mathbf{M}_{m \times n}$,

$\text{Sym}(n)$ denotes the space of real symmetric $n \times n$ -matrices,

$\Delta(n)$ denotes the vector space of all upper triangular $n \times n$ -matrices.

Definition 2.1 (see [C2]). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz mapping in a neighborhood U of x_0 in \mathbb{R}^n , i.e. there exists a constant $K > 0$ such that

$$\|f(x) - f(y)\| \leq K\|x - y\|, \text{ for all } x, y \in U.$$

Then we denote $L(f) \leq K$.

By Rademacher's theorem (see [F]), a Lipschitz mapping on a subset U of \mathbb{R}^n is differentiable almost everywhere. The Jacobian matrix of the partial derivatives of f at x , when it exists, is denoted by $Jf(x)$. The **generalized Jacobian** of f at x_0 , denoted by $\partial f(x_0)$, is the convex hull of all matrices M of the form

$$M = \lim_{i \rightarrow \infty} Jf(x_i),$$

where (x_i) converges to x_0 and f is differentiable at x_i for each i .

For $p \leq \min(m, n)$, we denote

$$\partial_{p \times p} f(x_0) = \{M_1 \in \mathbf{M}_{p \times p} : \text{there exists } M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in \partial f(x_0)\}.$$

$\partial f(x_0)$ is said to be of **maximal rank** if every M in $\partial f(x_0)$ has the maximal rank.

$\partial f(x_0)$ is said to be of **rank** p if every M in $\partial f(x_0)$ has rank p .

Definition 2.2. Let $f : U \rightarrow \mathbb{R}^m$ be a differentiable mapping of class C^k , $k \geq 1$, on an open subset U of \mathbb{R}^n . Then the C^k -norm of f is defined by

$$\|f\|_{C^k} = \max_{1 \leq p \leq k} \sup_{x \in U} \|D^p f(x)\|.$$

In the next sections we have to estimate the C^k -norm of compositions and inverses, to this aim we prepare the following two lemmas.

Lemma 2.3. *Let $f : U \rightarrow V$ and $g : V \rightarrow \mathbb{R}^p$ be differentiable mapping of class C^k , $k \geq 1$, on open subsets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$. Then*

$$\|g \circ f\|_{C^k} \leq (1^k + 2^k + \cdots + k^k) \|g\|_{C^k} \max(\|f\|_{C^k}, \|f\|_{C^k}^k).$$

In this paper, we denote

$$E(K_f, K_g, k) = (1^k + 2^k + \cdots + k^k) K_g \max(K_f, K_f^k).$$

Proof. By the Higher Order Chain Rule (see [A-M-R]), for $p \leq k$, we get the following estimation

$$\begin{aligned} \|D^p(g \circ f)(x)\| &\leq \sum_{i=1}^p \sum_{j_1+\cdots+j_i=p} \frac{p!}{j_1! \cdots j_i!} \|D^i g(f(x))\| \|D^{j_1} f(x)\| \cdots \|D^{j_i} f(x)\| \\ &\leq \sum_{i=1}^p \sum_{j_1+\cdots+j_i=p} \frac{p!}{j_1! \cdots j_i!} \|g\|_{C^k} \|f\|_{C^p}^i \\ &\leq \sum_{i=1}^p i^p \|g\|_{C^k} \max(\|f\|_{C^k}, \|f\|_{C^k}^p). \end{aligned}$$

So $\|g \circ f\|_{C^k} \leq (1^k + 2^k + \cdots + k^k) \|g\|_{C^k} \max(\|f\|_{C^k}, \|f\|_{C^k}^k)$. \square

Lemma 2.4. *Let $\varphi : U \rightarrow V$ be a C^k diffeomorphism between open subsets U, V of \mathbb{R}^n , $k \geq 1$. Then we have the estimation*

$$\|\varphi^{-1}\|_{C^k} \leq EI(\|\varphi\|_{C^k}, \|D\varphi^{-1}\|, k),$$

where EI is constructed by the following recurrent method:

Let $M_0 = E(\|\varphi\|_{C^k}, \max_{0 \leq p \leq k-1} p! \|D\varphi^{-1}\|^{p+1}, k-1)$, $M_1 = \|D\varphi^{-1}\|$, and $M_p = E(M_{p-1}, M_0, p-1)$, for $p = 2, \dots, k$. Then $EI(K, L, k) = M_k$.

Proof. Let

$$\begin{aligned} \text{Inv} : \text{Gl}(n) &\rightarrow \text{Gl}(n) \\ M &\mapsto \text{Inv}(M) = M^{-1}. \end{aligned}$$

Since $D\varphi^{-1} = \text{Inv} \circ D\varphi \circ \varphi^{-1}$, using Lemma 2.3, we have the recurrent inequalities

$$\|\varphi^{-1}\|_{C^p} = \max(\|D\varphi^{-1}\|, \|D\varphi^{-1}\|_{C^{p-1}}) \leq E(\|\varphi^{-1}\|_{C^{p-1}}, \|\text{Inv} \circ D\varphi\|_{C^{p-1}}, p-1), \text{ for } p \geq 2.$$

First, we estimate $\|\text{Inv} \circ D\varphi\|_{C^{k-1}}$. From $D^p \text{Inv}(M)(\delta M) = p!(-1)^p (M^{-1} \delta M)^p M^{-1}$, we get $\|D^p \text{Inv}(M)\| \leq p! \|M^{-1}\|^{p+1}$. Therefore, using the notations $K = \|\varphi\|_{C^k}$, $L = \|D\varphi^{-1}\|$, we have

$$\|D^p \text{Inv}(D\varphi(x))\| \leq p! \|(D\varphi(x))^{-1}\|^{p+1} \leq p! L^{p+1}.$$

Using Lemma 2.3, we get

$$\|\text{Inv} \circ D\varphi\|_{C^{k-1}} \leq E(K, \max_{1 \leq p \leq k-1} p! L^{p+1}, k-1) = M_0.$$

Let

$$M_1 = L, M_p = E(M_{p-1}, M_0, p-1), p = 2, \dots, k.$$

From $\|\varphi^{-1}\|_{C^1} = \|D\varphi^{-1}\| \leq M_1$, using Lemma 2.3 and recurrence, we have

$$\|\varphi^{-1}\|_{C^k} \leq M_k = EI(K, L, k).$$

□

Definition 2.5 (see [G-L]). Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear mapping. Then there exist $\sigma_1(L) \geq \dots \geq \sigma_r(L) > 0$, where $r = \text{rank} L$, so that $L(\mathbf{B}^n)$ is an r -dimensional ellipsoid of semi-axes $\sigma_1(L) \geq \dots \geq \sigma_r(L)$. Set $\sigma_0(L) = 1$ and $\sigma_{r+1}(L) = \dots = \sigma_m(L) = 0$, when $r < m$. We call $\sigma_0(L), \dots, \sigma_m(L)$ the **singular values** of L .

Remark 2.6.

- (i) $\sigma_1(L) = \|L\|$, $\sigma_m(L) = \min_{\|x\|=1} \|Lx\|$.
- (ii) If $\lambda \in \mathbb{R}$ is an eigenvalue of L , then $\sigma_m(L) \leq |\lambda| \leq \sigma_1(L)$.

Definition 2.7. A C^k function $f : U \rightarrow \mathbb{R}$ on an open subset U of \mathbb{R}^n , $k \geq 2$, is called **Morse** if for every critical point x of f , i.e. $Df(x) = 0$, the Hessian $Hf(x)$ is nondegenerate, i.e. $\sigma_n(Hf(x)) > 0$.

3. THE INVERSE, IMPLICIT AND RANK THEOREMS FOR LIPSCHITZ MAPPINGS

In this section, we present quantitative forms of the Inverse, Implicit and Rank Theorems for Lipschitz mappings, and give some explicit bounds in the smooth case.

Theorem 3.1 (Inverse Mapping Theorem). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz mapping. Suppose that $\partial f(x_0)$ is of maximal rank. Set*

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|},$$

and r be chosen so that $L(f) \leq K$ and

$$\partial f(x) \subset \partial f(x_0) + \delta \mathcal{B}_{n \times n}, \text{ when } x \in \mathbf{B}_r^n(x_0).$$

Then $f : \mathbf{B}_{\rho_1}^n(x_0) \rightarrow f(\mathbf{B}_{\rho_1}^n(x_0))$ is a homeomorphism, where $\rho_1 = \frac{r\delta}{2K}$, $f(\mathbf{B}_{\rho_1}^n(x_0))$ contains $\mathbf{B}_{\rho_2}^n(f(x_0))$, where $\rho_2 = \frac{r\delta}{2}$, and $L(f^{-1}) \leq \frac{1}{\delta}$ on $\mathbf{B}_{\rho_2}^n(f(x_0))$.

Proof. See [C1] and [P].

□

Remark 3.2. We make some comments on the pair (δ, r) of the theorem that are often used latter.

Let $\Sigma = \{A \in M_{n \times n} : \det A = 0\}$. Then by the Eckart-Young equality (see [G-L]), we have

$$\frac{1}{\|M^{-1}\|} = d(M, \Sigma), \text{ for every } M \text{ in } M_{n \times n} \setminus \Sigma.$$

Hence,

$$\delta = \frac{1}{2} \inf_{M_0 \in \partial f(x_0)} \frac{1}{\|M_0^{-1}\|} = \frac{1}{2} d(\partial f(x_0), \Sigma).$$

In words, δ is half the distance from the generalized Jacobian of f at x_0 to the singular locus Σ . Note that if $\delta' \leq \delta$, then the theorem is also true when δ is replaced by δ' .

By the upper semicontinuous property of the generalized Jacobian (see [C2]), for every $\delta > 0$ there exists $r > 0$, such that

$$\partial f(x) \subset \partial f(x_0) + \delta \mathcal{B}_{n \times n}, \text{ when } x \in \mathbf{B}_r^n(x_0).$$

So the quantity r reflects the rate of variation of the generalized Jacobian of f in a neighborhood of x_0 . If $r' \leq r$, then the theorem is also true when r is replaced by r' .

Using Lemma 2.4, we have the following corollary.

Corollary 3.3. With the assumptions and notations of Theorem 3.1, and in addition f is a C^k mapping, $k \geq 2$, and $\|f\|_{C^k} \leq K$. Then we can choose

$$\delta = \frac{1}{2\|Df(0)^{-1}\|}, \text{ and } r = \frac{\delta}{K}.$$

Moreover f^{-1} is also in class C^k , $\|Df^{-1}\| \leq \frac{1}{\delta}$, and

$$\|f^{-1}\|_{C^k} \leq EI(K, \frac{1}{\delta}, k).$$

Remark 3.4. If $F : U \times V \rightarrow \mathbb{R}^n$ be a Lipschitz mapping in a neighborhood of (x_0, y_0) in $\mathbb{R}^m \times \mathbb{R}^n$, then the generalized Jacobian of F at (x_0, y_0) satisfies

$$\partial F(x_0, y_0) \subset \left\{ \begin{pmatrix} M_1 & M_2 \end{pmatrix} : M_1 \in \partial_1 F(x_0, y_0), M_2 \in \partial_2 F(x_0, y_0) \right\},$$

where $\partial_1 F(x_0, y_0)$ and $\partial_2 F(x_0, y_0)$ are the generalized Jacobians of $F(\cdot, y_0) : U \rightarrow \mathbb{R}^n$ and $F(x_0, \cdot) : V \rightarrow \mathbb{R}^n$ at (x_0, y_0) , respectively.

Theorem 3.5 (Implicit Function Theorem). *Let $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz mapping in a neighborhood of (x_0, y_0) . Suppose that $\partial_2 F(x_0, y_0)$ is of maximal rank and $F(x_0, y_0) = 0$. Set*

$$\delta = \frac{1}{2} \inf_{M_2 \in \partial_2 F(x_0, y_0)} \frac{1}{(1 + (1 + K)^2 \|M_2^{-1}\|^2)^{\frac{1}{2}}},$$

and r be chosen so that $L(F) \leq K$ and

$$\partial F(x, y) \subset \partial F(x_0, y_0) + \delta \mathcal{B}_{n \times (m+n)}, \text{ when } (x, y) \in \mathbf{B}_r^{m+n}((x_0, y_0)).$$

Then there exists a Lipschitz mapping $g : \mathbf{B}_\rho(x_0) \rightarrow \mathbb{R}^n$, where $\rho = \frac{r\delta}{2(K+1)}$, and $L(g) \leq \frac{K}{\delta}$, such that

$$g(x_0) = y_0, \text{ and } F(x, g(x)) = 0, \text{ when } x \in \mathbf{B}_\rho(x_0).$$

Proof. (c.f. [Pa],[P]).

Set $f(x, y) = (x, F(x, y))$. Then f is Lipschitz with $L(f) \leq K + 1$ and

$$\partial f(x_0, y_0) \subset \left\{ \begin{pmatrix} I_m & 0 \\ M_1 & M_2 \end{pmatrix} : M_1 \in \partial_1 F(x_0, y_0), M_2 \in \partial_2 F(x_0, y_0) \right\}.$$

Since $\partial_2 F(x_0, y_0)$ is of maximal rank, $\partial f(x_0, y_0)$ is of maximal rank.

For $M = \begin{pmatrix} I_m & 0 \\ M_1 & M_2 \end{pmatrix} \in \partial f(x_0, y_0)$, we have $M^{-1} = \begin{pmatrix} I_m & 0 \\ -M_2^{-1}M_1 & M_2^{-1} \end{pmatrix}$.

Therefore,

$$\|M^{-1}\| = \sup_{\|x\|^2 + \|y\|^2 = 1} (\|x\|^2 + \|M_2^{-1}M_1x - M_2^{-1}y\|^2)^{\frac{1}{2}} \leq (1 + (K + 1)^2 \|M_2^{-1}\|^2)^{\frac{1}{2}},$$

and hence

$$\frac{1}{\|M^{-1}\|} \geq \frac{1}{(1 + (K + 1)^2 \|M_2^{-1}\|^2)^{\frac{1}{2}}}.$$

Since

$$\Delta = \frac{1}{2} \inf_{M \in \partial f(x_0, y_0)} \frac{1}{\|M^{-1}\|} \geq \delta,$$

by the supposition, we have

$$\partial f(x, y) \subset \partial f(x_0, y_0) + \Delta \mathcal{B}_{n \times n}, \text{ when } (x, y) \in \mathbf{B}_r^{m+n}((x_0, y_0)).$$

By Theorem 3.1, f is locally invertible, with $f^{-1}(x, z) = (x, h(x, z))$, $(x, z) \in \mathbf{B}_\rho^{n+m}(x_0, 0)$, where $\rho = \frac{r\delta}{2(K+1)}$. Let $g(x) = h(x, 0)$, $x \in \mathbf{B}_\rho^m(x_0)$. Then g is Lipschitz, $g(x_0) = y_0$ and $F(x, g(x)) = 0$.

Moreover, when F is differentiable at $(x, g(x))$, we have

$$\begin{aligned} \|Dg(x)\| &= \left\| - \left(\frac{\partial F}{\partial y} \right)^{-1} \frac{\partial F}{\partial x}(x, g(x)) \right\| \\ &\leq \sup_{M_2 \in \partial_2 F(x, y), (x, y) \in \mathbf{B}_r^{m+n}(x_0, y_0)} \|M_2^{-1}\| K \\ &\leq \sup_{M_2 \in \partial_2 F(x_0, y_0) + \delta \mathcal{B}_{n \times n}} \|M_2^{-1}\| K \leq \frac{1}{\delta} K. \end{aligned}$$

So $L(g) \leq \frac{K}{\delta}$. □

Corollary 3.6. With the assumptions and notations of Theorem 3.5, and in addition F is a C^k mapping, $k \geq 2$, and $\|F\|_{C^k} \leq K$. Then we can choose

$$\delta = \frac{1}{2(1 + (1 + K)^2 \|\frac{\partial F}{\partial y}(0, 0)^{-1}\|^2)^{\frac{1}{2}}}, \text{ and } r = \frac{\delta}{K}.$$

Moreover g is also in class C^k , $\|Dg\| \leq \frac{K}{\delta}$, and

$$\|g\|_{C^k} \leq C(K, k) = 2^{k-1} EI(K, \frac{K}{\delta}, k-1)K.$$

Proof. From $F(x, g(x)) = 0$, we have $Dg(x) = - \left(\frac{\partial F}{\partial y} \right)^{-1} \frac{\partial F}{\partial x}(x, g(x))$.

By Theorem 3.5, $\|g\|_{C^1} = \|Dg\| \leq \frac{K}{\delta}$.

Applying the Higher Order Leibnitz Rule (see [A-M-R]), we have

$$\begin{aligned} \|g\|_{C^k} &= \max(\|Dg\|, \|Dg\|_{C^{k-1}}) \\ &\leq \sum_{i=0}^{k-1} \binom{k-1}{i} \left\| \left(\frac{\partial F}{\partial y} \right)^{-1} \right\|_{C^{k-1}} \left\| \frac{\partial F}{\partial x} \right\|_{C^{k-1}} \\ &\leq 2^{k-1} \left\| \left(\frac{\partial F}{\partial y} \right)^{-1} \right\|_{C^{k-1}} K. \end{aligned}$$

To estimate $\left\| \left(\frac{\partial F}{\partial y} \right)^{-1} \right\|_{C^{k-1}}$, we use Lemma 2.4 to get

$$\left\| \left(\frac{\partial F}{\partial y} \right)^{-1} \right\|_{C^{k-1}} \leq EI(\left\| \frac{\partial F}{\partial y} \right\|_{C^{k-1}}, \|D \left(\frac{\partial F}{\partial y} \right)^{-1}\|, k-1) \leq EI(K, \frac{K}{\delta}, k-1).$$

From this estimation, we get

$$\|g\|_{C^k} \leq 2^{k-1} EI(K, \frac{K}{\delta}, k-1) K.$$

□

Theorem 3.7 (Rank Theorem). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz mapping in a neighborhood U of $x_0 \in \mathbb{R}^n$ with $L(f) \leq K$. Suppose that $\partial f(x)$ is of rank p for all $x \in U$ and that $\partial_{p \times p} f(x_0)$ is of rank p . Set*

$$\delta = \frac{1}{2} \inf_{M_1 \in \partial_{p \times p} f(x_0)} \frac{1}{(1 + (1 + K)^2 \|M_1^{-1}\|^2)^{\frac{1}{2}}},$$

and r be chosen so that $\mathbf{B}_r^n(x_0) \subset U$ and

$$\partial f(x) \subset \partial f(x_0) + \delta \mathcal{B}_{n \times n}, \text{ for all } x \in \mathbf{B}_r^n(x_0).$$

Then there exist homeomorphisms

$$\varphi : \mathbf{B}_{\rho_1}^n(x_0) \rightarrow \varphi(\mathbf{B}_{\rho_1}^n(x_0)), \text{ and } \psi : \mathbf{B}_{\rho_2}^m(f(x_0)) \rightarrow \psi(\mathbf{B}_{\rho_2}^m(f(x_0))),$$

where $\rho_1 = \frac{r\delta}{2(K+1)}$, $\rho_2 = \frac{r\delta}{2}$, $L(\varphi) \leq \frac{1}{\delta}$, and $L(\psi) \leq 1 + \frac{K}{\delta}$,

such that $\varphi(\mathbf{B}_{\rho_1}^n(x_0)) \supset \mathbf{B}_{\rho_2}^n(\varphi(x_0))$, $\varphi^{-1} : \mathbf{B}_{\rho_2}^n(\varphi(x_0)) \rightarrow \mathbb{R}^n$, $L(\varphi^{-1}) \leq \frac{1}{K+1}$, and

$$\psi \circ f \circ \varphi^{-1}(z_1, \dots, z_n) = (z_1, \dots, z_p, 0, \dots, 0).$$

Proof. Without loss of generality we can assume $x_0 = 0$, $f(0) = 0$.

Let $\varphi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$, be defined by

$$\varphi(x_1, \dots, x_n) = (f_1(x), \dots, f_p(x), x_{p+1}, \dots, x_n).$$

Then $L(\varphi) \leq K + 1$, and each $M \in \partial\varphi(0)$ has the form

$$M = \begin{pmatrix} M_1 & M_2 \\ 0 & I \end{pmatrix}, \text{ where } M_1 \in \partial_{p \times p} f(0).$$

Hence $\partial\varphi(0)$ is of maximal rank. By an estimation in the proof of Theorem 3.5, we have

$$\|M^{-1}\| \leq (1 + (1 + K)^2 \|M_1^{-1}\|^2)^{\frac{1}{2}}.$$

So $\delta_1 = \frac{1}{2} \inf_{M \in \partial\varphi(0)} \frac{1}{\|M^{-1}\|} \geq \delta = \frac{1}{2} \inf_{M_1 \in \partial_{p \times p} f(x_0)} \frac{1}{(1 + (1 + K)^2 \|M_1^{-1}\|^2)^{\frac{1}{2}}}$. From this inequality and the assumption, we have

$$\partial\varphi(x) \subset \partial\varphi(0) + \delta_1 \mathcal{B}_{n \times n}, \text{ for all } x \in \mathbf{B}_r^n.$$

Hence we can apply Theorem 3.1, to conclude that $\varphi : \mathbf{B}_{\rho_1} \rightarrow \varphi(\mathbf{B}_{\rho_1})$ is a homeomorphism, where $\rho_1 = \frac{r\delta}{2(K+1)}$, $\varphi(\mathbf{B}_{\rho_1}) \supset \mathbf{B}_{\rho_2}$, where $\rho_2 = \frac{r\delta}{2}$, and $L(\varphi^{-1}) \leq \frac{1}{\delta}$. Let

$$g = f \circ \varphi^{-1} : \mathbf{B}_{\rho_2} \rightarrow \mathbb{R}^m.$$

Then $L(g) \leq L(f)L(\varphi^{-1}) \leq \frac{K}{\delta}$, and $g(z_1, \dots, z_n) = (z_1, \dots, z_p, g_{p+1}(z), \dots, g_m(z))$. Therefore, each $M' \in \partial g(z)$ is of the form

$$M' = \begin{pmatrix} I & 0 \\ M_1 & M_2 \end{pmatrix}.$$

Since $\partial g(z)$ is of rank p when $z \in \mathbf{B}_{\rho_2}$,

$$(3.1) \quad M_2 = 0.$$

Let $\psi : \mathbf{B}_{\rho_2} \rightarrow \mathbb{R}^m$, be given by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \\ y_{p+1} \\ \vdots \\ y_m \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_p \\ y_{p+1} - g_{p+1}(y_1, \dots, y_p, 0, \dots, 0) \\ \vdots \\ y_m - g_m(y_1, \dots, y_p, 0, \dots, 0) \end{pmatrix}$$

Then $L(\psi) \leq 1 + L(g) \leq 1 + \frac{K}{\delta}$, and each $M'' \in \partial\psi(y)$ is of the form

$$M'' = \begin{pmatrix} I & 0 \\ ? & I \end{pmatrix}$$

By Theorem 3.1, ψ is locally invertible. It is easy to see that ψ is injective. So ψ is a homeomorphism from \mathbf{B}_{ρ_2} onto its image.

Because of (3.1), $g_j(z_1, \dots, z_m) - g_j(z_1, \dots, z_p, 0, \dots, 0) = 0$, for every $z \in \mathbf{B}_{\rho_2}$ and $j > p$. Therefore $\psi \circ f \circ \varphi^{-1} = \psi \circ g$ is represented by

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_p, 0, \dots, 0).$$

□

Corollary 3.8. With the assumptions and notations of Theorem 3.7, and in addition f is of class C^k , $k \geq 2$, and $\|f\|_{C^k} \leq K$. Then we can choose

$$\delta = \frac{1}{2(1 + (1 + K)^2 \|J_{p \times p} f(x_0)\|^2)^{\frac{1}{2}}}, \text{ and } r = \frac{K}{\delta}.$$

Moreover φ and ψ are of class C^k , $\|\varphi\|_{C^k} \leq K + 1$, $\|D\psi\| \leq \frac{K}{\delta} + 1$, and

$$\|\psi\|_{C^k} \leq C(K, \delta, k) = E(EI(K + 1, \frac{1}{\delta}, k), K, k).$$

Proof. It suffices to estimate $\|\varphi\|_{C^k}$ and $\|\psi\|_{C^k}$.

By the definition of φ , we have $\|\varphi\|_{C^k} \leq \|f\|_{C^k} + 1 \leq K + 1$.

From $\|D\varphi^{-1}\| \leq \frac{1}{\delta}$, using Lemma 2.4, we have $\|\varphi^{-1}\|_{C^k} \leq EI(K + 1, \frac{1}{\delta}, k)$.

From this inequality, using Lemma 2.3, we get

$$\|g\|_{C^k} = \|f \circ \varphi^{-1}\|_{C^k} \leq E(EI(K + 1, \frac{1}{\delta}, k), K, k).$$

Finally, by the definition of ψ , $\|\psi\|_{C^k} \leq \|g\|_{C^k} + 1$, and hence

$$\|D\psi\| \leq \|Dg\| + 1 \leq \frac{K}{\delta} + 1, \text{ and } \|\psi\|_{C^k} \leq E(EI(K + 1, \frac{1}{\delta}, k), K, k) + 1.$$

□

4. SPLITTING LEMMA - MORSE FUNCTIONS.

In this section we give the quantitative versions of Splitting Lemma, Morse Lemma, the density and openness of Morse functions on a ball.

To prove the Splitting Lemma, we prepare the following lemma, which gives a quantitative form of diagonalization of matrix-valued mappings by upper triangular matrices.

Lemma 4.1. Let $\bar{B} : U \rightarrow \text{Sym}(n)$, be a C^k matrix-valued mappings, $k \geq 1$, on a open neighborhood U of 0 in \mathbb{R}^n . Suppose that $\|\bar{B}\|_{C^k} \leq \bar{K}$, and

$$\bar{B}(0) = D_0 = \text{diag}(\pm 1, \dots, \pm 1).$$

Let $\delta = \frac{1}{4(\bar{K} + 1)(\bar{K} + 2)(1 + (\bar{K} + 2)^2 n(n + 1))}$. Then there exists a C^k mapping

$$\mathcal{Q} : U_\delta \rightarrow \Delta(n), \text{ where } U_\delta = U \cap \mathbf{B}_\delta^n,$$

such that $\mathcal{Q}(0) = I_n$, $\bar{B}(x) = {}^t \mathcal{Q}(x) D_0 \mathcal{Q}(x)$, and

$$\|D\mathcal{Q}(x)\| \leq (\bar{K} + 1)(1 + (\bar{K} + 2)n(n + 1)), \|\mathcal{Q}\|_{C^k} \leq \bar{C}(\bar{K}, n, k),$$

where $\bar{C}(\bar{K}, n, k) = 2^{k-1}(\bar{K} + 1)EI(\bar{K} + 1, (\bar{K} + 1)(1 + (\bar{K} + 2)n(n + 1)), k - 1)$.

Proof. Let $s = \frac{n(n+1)}{2} = \dim \Delta(n) = \dim \text{Sym}(n)$.

Consider $F : U \times \Delta(n) \rightarrow \text{Sym}(n)$, $F(x, Q) = \overline{B}(x) - {}^t Q D_0 Q$.

Then $F \in C^k$, $F(0, I_n) = 0 \in \mathbf{M}_{n \times n}$, and

$$\frac{\partial F}{\partial Q}(0, I_n) : \Delta(n) \rightarrow \text{Sym}(n), \quad \frac{\partial F}{\partial Q}(0, I_n)(H) = -{}^t H D_0 - D_0 H.$$

Denote $H = (h_{ij})$, $G = (g_{ij}) = -({}^t H D_0 + D_0 H)$. Then

$$\begin{cases} h_{ij} &= -g_{ij}, i < j, \\ h_{ij} &= \pm \frac{1}{2} g_{ii}. \end{cases}.$$

Thus $\frac{\partial F}{\partial Q}(0, I_n)$ is invertible, and

$$\begin{aligned} \left\| \frac{\partial F}{\partial Q}(0, I_n) \right\| &= \sup_{\|H\|=1} \|{}^t H D_0 - D_0 H\| \leq \sup_{\|H\|=1} 2\|H\| \|D_0\| = 2, \\ \left\| \left(\frac{\partial F}{\partial Q}(0, I_n) \right)^{-1} \right\| &\leq \left\| \left(\frac{\partial F}{\partial Q}(0, I_n) \right)^{-1} \right\|_F \leq \sqrt{s} = \sqrt{\frac{n(n+1)}{2}}. \end{aligned}$$

We are using Implicit Function Theorem, so we estimate some numbers related to F . From

(4.1)

$$F(x + \Delta x, Q + H) - F(x, Q) = \overline{B}(x + \Delta x) - \overline{B}(x) - {}^t H D_0 Q - {}^t Q D_0 H - {}^t H Q D_0 H,$$

we have

$$\frac{\partial F}{\partial x}(x, Q) = D\overline{B}(x), \quad \frac{\partial F}{\partial Q}(x, Q)(H) = -{}^t H \overline{B}(x) Q - {}^t Q \overline{B}(x) H.$$

For $0 < r \leq \frac{\sqrt{1+2\bar{K}}}{\sqrt{2\bar{K}}}$, when $\|(x, Q)\| \leq r$, we have

$$\begin{aligned} \|DF(x, Q)\| &\leq (\|D\overline{B}(x)\|^2 + \sup_{\|H\|=1} \|{}^t H \overline{B}(x) Q + {}^t Q \overline{B}(x) H\|^2)^{\frac{1}{2}} \\ &\leq (\bar{K}^2 + 2\|\overline{B}(x)\|^2 \|Q\|^2)^{\frac{1}{2}} \leq \bar{K} \sqrt{1 + 2r^2} \leq \bar{K} + 1. \end{aligned}$$

So $L(F) \leq \bar{K} + 1$ on \mathbf{B}_r^{n+s} . To apply Theorem 3.5 to F , with

$$\delta_1 = \frac{1}{2(1 + (1 + (\bar{K} + 1)^2 s))^{\frac{1}{2}}},$$

we chose $r = \min(\frac{\sqrt{1+2\bar{K}}}{\sqrt{2\bar{K}}}, \frac{\delta_1}{\bar{K}+1}) = \frac{\delta_1}{\bar{K}+1}$ to have

$$\|DF(x, Q) - DF(0, I_n)\| < (\bar{K} + 1)r = \delta_1, \quad \text{when } \|(x, Q)\| < r.$$

According to Theorem 3.5 and Corollary 3.6, there exists a C^k mapping

$$\mathcal{Q} : U_\delta \rightarrow \Delta(n) \text{ in class } C^k, \quad U_\delta = U \cap \mathbf{B}_\delta^n,$$

where $\delta = \frac{r\delta_1}{2(\bar{K} + 2)} = \frac{1}{8(\bar{K} + 1)(\bar{K} + 2)(1 + (\bar{K} + 2)^2 s)}$, such that

$$\mathcal{Q}(0) = I_n, \quad F(x, \mathcal{Q}(x)) = \overline{B}(x) - {}^t \mathcal{Q}(x) D_0 \mathcal{Q}(x) = 0, \quad \text{and } \|D\mathcal{Q}(x)\| \leq \frac{\bar{K} + 1}{\delta_1}.$$

Moreover, from (4.1), we have

$$\begin{aligned} \frac{\partial^p F}{\partial x^p}(x, Q) &= D^p \bar{B}(x), \\ \frac{\partial F}{\partial Q}(x, Q)(\Delta x, \Delta Q) &= -{}^t \Delta Q D_0 Q - {}^t Q D_0 \Delta Q, \text{ and hence } \|\frac{\partial F}{\partial Q}\| \leq 2\|Q\| \leq 2, \\ \frac{\partial^2 F}{\partial Q^2}(x, Q)(\Delta x, \Delta Q) &= -{}^t \Delta Q D_0 \Delta Q, \text{ and hence } \|\frac{\partial^2 F}{\partial Q^2}(x, Q)\| \leq 1, \\ \frac{\partial^{|\alpha|+|\beta|} F}{\partial x^\alpha \partial Q^\beta}(x, Q) &= 0, \text{ when } \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^s, \alpha \neq 0 \neq \beta \text{ or } |\beta| \geq 3. \end{aligned}$$

So $\|F\|_{C^k} \leq \bar{K} + 2$. Using Corollary 3.6, we get

$$\|\mathcal{Q}\|_{C^k} \leq \bar{C}(\bar{K}, n, k) = 2^{k-1}(\bar{K} + 2)EI(\bar{K} + 2, \frac{\bar{K} + 2}{\delta_1}, k - 1).$$

□

Applying the above lemma and Implicit Function Theorem, we can get a quantitative form of Splitting Lemma.

Theorem 4.2 (Splitting Lemma). *Let $f : U \rightarrow \mathbb{R}$ be a C^k function on a neighborhood U of x_0 in \mathbb{R}^n , $k \geq 3$. Suppose that $\|f\|_{C^k} \leq K$, and*

$$Df(x_0) = 0, \text{ rank } D^2 f(x_0) = p.$$

Let

$$\delta = \frac{\sigma_p^{5/2}}{32(K+1)^{9/2}} \min(1, \frac{2\sigma_p^2}{3(p^2+p+1)}, \frac{\sigma_p^{3/2}}{2(p^2+p+1)}), \text{ where } \sigma_p = \sigma_p(D^2 f(x_0)).$$

Then there exists a C^{k-1} diffeomorphism

$$\varphi : \mathbf{B}_\delta^n \rightarrow \varphi(\mathbf{B}_\delta^n), \text{ with } \|D\varphi\| \leq \frac{32(K+1)^5}{\sigma_p^{5/2}},$$

such that

$$f \circ \varphi(x, y) = f(x_0) + \sum_{i=1}^p \pm x_i^2 + \alpha(y), \quad x = (x_1, \dots, x_p) \in \mathbb{R}^p, y \in \mathbb{R}^{n-p},$$

where α is of class C^k and $\alpha(0), D\alpha(0), D^2\alpha(0)$ vanish.

Moreover, there exists a constant $M(K, \sigma_p, k) > 0$, such that

$$\|\varphi\|_{C^{k-1}} \leq M(K, \sigma_p, k).$$

Proof. We can assume $x_0 = 0, f(x_0) = 0$, and can choose the coordinate system $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q, p+q = n, x = (x_1, \dots, x_p), y = (y_1, \dots, y_q)$ so that

$$A = H_1 f(0, 0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0, 0) \right)_{1 \leq i, j \leq p}$$

is of rank p . Note that if $\sigma_1 \geq \dots \geq \sigma_p > 0$ are the singular values of A , then $\frac{1}{\sigma_p} \geq \dots \geq \frac{1}{\sigma_1} > 0$ are the singular values of A^{-1} and

$$\frac{1}{\sigma_1} \leq \|A^{-1}\| \leq \frac{1}{\sigma_p}.$$

Step 1. Consider the equation: $\frac{\partial f}{\partial x}(x, y) = 0$.

We have $\frac{\partial f}{\partial x}(0, 0) = 0$. To apply the Implicit Function Theorem 3.5 to $\frac{\partial f}{\partial x}$, we will determine some numbers. Let

$$\delta' = \frac{1}{2(1 + (K + 1)^2 \frac{1}{\sigma_p^2})^{\frac{1}{2}}}, \text{ and } r' = \frac{\delta'}{K}.$$

Then, for $\|(x, y)\| < r'$,

$$\|D(\frac{\partial f}{\partial x})(x, y) - D(\frac{\partial f}{\partial x})(0, 0)\| < Kr' = \delta'.$$

So we can apply Theorem 3.5 and its corollary 3.6, to have a C^{k-1} mapping

$$g : \mathbf{B}_{\delta_1}^q \rightarrow \mathbb{R}^p,$$

where

$$\delta_1 = \frac{r'\delta'}{2(K + 1)} = \frac{1}{8K(K + 1)(1 + (K + 1)^2 \frac{1}{\sigma_p^2})},$$

such that

$$g(0) = 0, \quad \frac{\partial f}{\partial x}(g(y), y) = 0,$$

and

$$(4.2) \quad \|Dg\| \leq \frac{K}{\delta_1}, \quad \|g\|_{C^{k-1}} \leq 2^{k-2}EI(K, \frac{K}{\delta_1}, k-2)K = M_1(K, \sigma_p).$$

Let $\alpha(y) = f(g(y), y)$. Then $\alpha(0) = 0$, $D\alpha(0) = 0$, and

$$D\alpha(y) = \frac{\partial f}{\partial x}(g(y), y)Dg(y) + \frac{\partial f}{\partial y}(g(y), y) = \frac{\partial f}{\partial y}(g(y), y).$$

Since the mapping on the right side is of class C^{k-1} , $D\alpha$ is of class C^{k-1} , and hence α is of class C^k . Moreover, $\frac{\partial f}{\partial x}(g(y), y) \equiv 0$ and $\frac{\partial^2 f}{\partial y^2}(0, 0) = 0$ imply $D^2\alpha(0) = 0$.

Let $f_1(x, y) = f(x, y) - \alpha(y)$, $(x, y) \in \mathbf{B}_{\delta_1}^n$. We have

$$f_1(g(y), y) = 0, \quad \frac{\partial f_1}{\partial x}(g(y), y) = 0.$$

Step 2. Let $\varphi_1(x, y) = (x + g(y), y)$, $(x, y) \in \mathbf{B}_{\delta_1}^n$. Then φ_1 is a C^{k-1} diffeomorphism from $\mathbf{B}_{\delta_1}^n$ to its image, and from (4.2), we get

$$(4.3) \quad \|D\varphi_1\| \leq 1 + \|Dg\| \leq 1 + \frac{K}{\delta_1}, \quad \|\varphi_1\|_{C^{k-1}} \leq 1 + M_1(K, \sigma_p) = M_2(K, \sigma_p).$$

Let $f_2 = f_1 \circ \varphi_1$. Then f_2 is of class C^{k-1} , and

$$f_2(0, y) = 0, \quad \frac{\partial f_2}{\partial x}(0, y) = 0.$$

Note that

$$\frac{\partial f_2}{\partial x}(x, y) = \frac{\partial f_1}{\partial x}(x + g(y), y), \text{ and } \frac{\partial^2 f_2}{\partial x^2} = {}^t\frac{\partial \varphi_1}{\partial x} \frac{\partial^2 f_1}{\partial x^2} \frac{\partial \varphi_1}{\partial x} = \frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}.$$

Step 3. Let $Q_0 \in \text{Gl}(p)$ be the linear transformation so that

$${}^tQ_0 A Q_0 = D_0 = (\pm 1, \dots, \pm 1).$$

Moreover, choose $Q_0 = SU$, where U is an orthogonal matrix and S is a diagonal matrix, so that

$$\|Q_0\|^2 = \frac{1}{\sigma_p}, \|Q_0^{-1}\|^2 = \sigma_1 = \|A\| \leq K.$$

Let $B : \mathbf{B}_{\delta_1}^n \rightarrow \text{Sym}(p)$, defined by

$$B(x, y) = (b_{ij}(x, y))_{1 \leq i, j \leq p}, \text{ where } b_{ij}(x, y) = \int_0^1 \int_0^1 \frac{\partial^2 f_2}{\partial x_i \partial x_j}(stx, y) ds dt.$$

Then B is of class C^{k-1} , and

$$f_2(x, y) = {}^t x B(x, y) x, \text{ and } B(0, 0) = A = H_1 f(0, 0).$$

Set

$$\overline{B}(x, y) = {}^t Q_0 B(x, y) Q_0.$$

Then $\overline{B} : \mathbf{B}_{\delta_1}^n \rightarrow \text{Sym}(p) \in C^{k-1}$, $\overline{B}(0, 0) = D_0$, and $\|D\overline{B}\| \leq \bar{K} = \frac{K}{\sigma_p}$. According to Lemma 4.1, there exists a C^{k-1} mapping

$$\mathcal{Q} : \mathbf{B}_{\delta_2}^p \rightarrow \Delta(p),$$

where $\delta_2 = \min(\delta_1, \frac{1}{4(\bar{K} + 1)(\bar{K} + 2)(1 + (\bar{K} + 2)^2 p(p + 1))})$,
such that

$$\mathcal{Q}(0) = I_p, \overline{B}(x, y) = {}^t \mathcal{Q}(x) D_0 \mathcal{Q}(x), \|D\mathcal{Q}\| \leq (\bar{K} + 1)(1 + (\bar{K} + 2)^2 p(p + 1)),$$

and

$$(4.4) \quad \|\mathcal{Q}\|_{C^{k-1}} \leq \overline{C}(\bar{K}, p, k - 1) = M_3(K, \sigma_p).$$

Let $\varphi_2(x, y) = (\mathcal{Q}(x)Q_0^{-1}x, y)$, $(x, y) \in \mathbf{B}_{\delta_2}^n$.

We are applying the Inverse Mapping Theorem 3.1 to φ_2 . So we have to calculate to determine the pair (δ, r) (see Remark 3.2) and some numbers. First we have

$$D\varphi_2(x, y)(h, e) = (\mathcal{Q}(x)Q_0^{-1}h + D\mathcal{Q}(x)hQ_0^{-1}x, e).$$

Hence

$$D\varphi_2(0, 0)(h, e) = (Q_0^{-1}h, e), \text{ and } D\varphi_2(0, 0)^{-1}(H, e) = (Q_0 H, e).$$

Thus

$$\frac{1}{\|D\varphi_2(0, 0)^{-1}\|} = \frac{1}{\sup_{\|H\|=1} (\|Q_0 h\|^2 + 1)^{\frac{1}{2}}} = \frac{1}{\sqrt{\|Q_0\|^2 + 1}} = \sqrt{\frac{\sigma_p}{1 + \sigma_p}}.$$

So we get $\delta_3 = \frac{1}{2}\sqrt{\frac{\sigma_p}{1+\sigma_p}}$. Let $r_3 = \frac{\delta_3}{2(\bar{K}+1)(1+(\bar{K}+2)^2p(p+1))\sqrt{\bar{K}}}$. Applying the Mean Value Theorem and (4.4), when $\|(x, y)\| < r_3$, we have

$$\begin{aligned} \|D\varphi_2(x, y) - D\varphi_2(0, 0)\| &= \sup_{\|h\|=1} \|\mathcal{Q}(x)Q_0^{-1}h + D\mathcal{Q}(x)hQ_0^{-1}x - \mathcal{Q}(0)Q_0^{-1}h\| \\ &\leq \|\mathcal{Q}(x) - \mathcal{Q}(0)\|\|Q_0^{-1}\| + \|D\mathcal{Q}(x)\|\|Q_0^{-1}\|\|x\| \\ &< 2\|D\mathcal{Q}\|r_3\|Q_0^{-1}\| \\ &\leq 2(\bar{K}+1)(1+(\bar{K}+2)^2p(p+1))\sqrt{\bar{K}}r_3 = \delta_3. \end{aligned}$$

Now applying the Inverse Mapping Theorem 3.1 to φ_2 , we have $\varphi_2^{-1} : \mathbf{B}_\delta^n \rightarrow \mathbb{R}^n$, where

$$(4.5) \quad \delta = \frac{\min(\delta_2, r_3)\delta_3}{2}, \text{ and } \|D\varphi_2^{-1}\| \leq \frac{1}{\delta_1} = 2\sqrt{\frac{1+\sigma_p}{\sigma_p}}.$$

Let

$$\varphi = \varphi_1 \circ \varphi_2^{-1} : \mathbf{B}_\delta^n \rightarrow \varphi(\mathbf{B}_\delta^n).$$

Note that we used $L(\varphi_2) = (\bar{K}+1)(1+(\bar{K}+2)^2p(p+1)) > 1$, so, by Theorem 3.1, $\varphi_2^{-1}(\mathbf{B}_\delta^n) \subset \mathbf{B}_{\delta_1}^n$ (the domain of φ_1). By the C^{k-1} coordinate transformation φ , we have

$$\begin{aligned} f \circ \varphi(x, y) &= f_1(\varphi_1(\varphi_2^{-1}(x, y)) + \alpha(y)) \\ &= f_2 \circ \varphi_2^{-1}(x, y) + \alpha(y) \\ &= \sum_{i=1}^p \pm x_i^2 + \alpha(y). \end{aligned}$$

Using (4.3) (4.5) and $\sigma_p \leq K$, we can easily get the following estimate

$$\begin{aligned} \|D\varphi\| \leq \|D\varphi_1\|\|D\varphi_2^{-1}\| &\leq (1 + 8K^2(K+1)(1 + \frac{(K+1)^2}{\sigma_p^2}))2\sqrt{\frac{1+\sigma_p}{\sigma_p}} \\ &< \frac{32}{\sigma_p^{5/2}}(K+1)^5. \end{aligned}$$

Moreover, using the Leibnitz Rule, we have

$$\begin{aligned} \|\varphi_2\|_{C^{k-1}} &\leq \|\mathcal{Q}Q_0^{-1}\|_{C^{k-1}} + 1 \\ &\leq 2^{k-1}\|\mathcal{Q}\|_{C^{k-1}}\|Q_0^{-1}\| + 1 \\ &\leq 2^{k-1}M_3(K, \sigma_p)\sqrt{\bar{K}} + 1 = M_4(K, \sigma_p). \end{aligned}$$

From this estimation, (4.3) (4.5) and using Lemmas 2.3, 2.4, we get

$$\begin{aligned} \|\varphi\|_{C^{k-1}} &= \|\varphi_1 \circ \varphi_2^{-1}\|_{C^{k-1}} \\ &\leq E(\|\varphi_2^{-1}\|_{C^{k-1}}, \|\varphi_1\|_{C^{k-1}}, k-1) \\ &\leq E(EI(M_4, 2\sqrt{\frac{1+\sigma_p}{\sigma_p}}, k-1), M_2, k-1) = M(K, \sigma_p, k). \end{aligned}$$

Step 4. To avoid the complicated formula for δ we make some elementary estimates.

Keeping track of the numbers during the proof, from 4.5, we have

$$\delta = \frac{1}{8}\delta_3 \min(a, b, c),$$

where

$$a = \frac{1}{2K(K+1)(1 + \frac{(K+1)^2}{\sigma_p^2})},$$

$$b = \frac{1}{(\frac{K}{\sigma_p} + 1)(\frac{K}{\sigma_p} + 2)(1 + (\frac{K}{\sigma_p} + 2)^2 p(p+1))},$$

$$c = \frac{\sqrt{\sigma_p}}{\sqrt{1 + \sigma_p(\frac{K}{\sigma_p} + 1)(1 + (\frac{K}{\sigma_p} + 2)^2 p(p+1))}\sqrt{K}}.$$

Use $\sigma_p \leq K$ to get

$$\begin{aligned} a &= \frac{\sigma_p^2}{K(K+1)(\sigma_p^2 + (K+1)^2)} > \frac{\sigma_p^2}{4(K+1)^4}, \\ b &= \frac{\sigma_p^4}{(K+\sigma_p)(K+2\sigma_p)(\sigma_p^2 + (K+2\sigma_p)^2 p(p+1))} > \frac{\sigma_p^4}{6(K+1)^4(p^2+p+1)}, \\ c &= \frac{\sqrt{\sigma_p}\sigma_p^3}{\sqrt{1+\sigma_p(K+\sigma_p)(\sigma_p^2 + (K+2\sigma_p)^2 p(p+1))}\sqrt{K}} > \frac{\sqrt{\sigma_p}\sigma_p^3}{2(K+1)^4(p^2+p+1)}. \end{aligned}$$

So

$$\begin{aligned} \delta &> \frac{1}{8} \sqrt{\frac{\sigma_p}{\sigma_p+1}} \frac{\sigma_p^2}{4(K+1)^4} \min(1, \frac{2\sigma_p^2}{3(p^2+p+1)}, \frac{\sigma_p^{3/2}}{2(p^2+p+1)}) \\ &> \frac{\sigma_p^{5/2}}{32(K+1)^{9/2}} \min(1, \frac{2\sigma_p^2}{3(p^2+p+1)}, \frac{\sigma_p^{3/2}}{2(p^2+p+1)}). \end{aligned}$$

We reduce the radius of the domain of φ to the last number to use in the statement of the theorem. \square

Theorem 4.3 (Morse Lemma). *With the assumptions and notations of Theorem 4.2, when $p = n$ we have*

$$f \circ \varphi(x_1, \dots, x_n) = f(x_0) + \sum_{i=1}^n \pm x_i^2.$$

Applying the quantitative Morse-Sard Theorem (see [Y1] or [Y-C]) and the Inverse Mapping Theorem, we give here a version for the density of Morse functions on a ball (c.f. [Y2, Th. 4.1, Th. 6.1]).

Theorem 4.4. *Fix $k \geq 3$. Let $f_0 : \overline{\mathbf{B}}^n \rightarrow \mathbb{R}$ be a C^k -function with $\|f_0\|_{C^k} \leq K$. Then for any given $\varepsilon > 0$, we can find h with $\|h\|_{C^k} \leq \varepsilon$ and the positive functions $\psi_1, \psi_2, \psi_3, d, M, N, \eta$ depending on K and ε , such that $f = f_0 + h$ satisfies the following conditions:*

- (i) *At each critical point x_i of f , $\sigma_n(Hf(x_i)) \geq \psi_1(K, \varepsilon)$.*
- (ii) *For any two different critical points x_i and x_j of f , $\|x_i - x_j\| \geq d(K, \varepsilon)$.
Consequently, the number of critical points does not exceed $N(K, \varepsilon)$.*
- (iii) *For any two different critical points x_i and x_j of f , $|f(x_i) - f(x_j)| \geq \psi_2(K, \varepsilon)$.*
- (iv) *For each critical point x_i of f , there exists a C^{k-1} coordinate transformation $\varphi : \mathbf{B}_\delta^n(x_i) \rightarrow \mathbb{R}^n$ such that*

$$f \circ \varphi^{-1}(y_1, \dots, y_n) = y_1^2 + \dots + y_l^2 - y_{l+1}^2 - \dots - y_n^2 + \text{const},$$

where $\delta = \psi_3(K, \varepsilon)$ and $\|\varphi\|_{C^{k-1}} \leq M(K, \varepsilon)$.

(v) If $\|Df(x)\| \leq \eta(K, \varepsilon)$, then $x \in \mathbf{B}_\delta^n(x_i)$, with x_i is a critical point of f .

Proof. In [L-P], our proof of the theorem needs some corrections. Moreover, we can apply the Splitting Lemma 4.2 to get an alternative proof of (iv) in that paper with more explicit estimations for δ and M . For these reasons, we make some improvements in detail in this present paper.

(i) We are applying the results of Chapter 9 [Y-C] to Df_0 . For $\gamma > 0$, denote $\bar{\gamma} = (\lambda_1, \lambda_2, \dots, \lambda_n) = (K, K, \dots, \gamma)$. Then, by definition, the set of $\bar{\gamma}$ -critical points and the set of $\bar{\gamma}$ -critical values of f are

$$\begin{aligned} \Sigma(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) &= \{x \in \bar{\mathbf{B}}^n : \sigma_i(D(Df_0)(x)) \leq \lambda_i, i = 1, \dots, n\} \\ &= \{x \in \bar{\mathbf{B}}^n : \sigma_n(Hf_0(x)) \leq \gamma\}, \text{ and} \\ \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) &= f(\Sigma(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n)). \end{aligned}$$

For a relatively compact subset A of \mathbb{R}^n , and $r > 0$, denoted by $M(r, A)$ the minimal number of balls of radius r in \mathbb{R}^n , covering A .

Let $\varepsilon > 0$. Applying Theorem 9.6 of [Y-C], when $0 < r < \varepsilon$,

$$\begin{aligned} M(r, \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) \cap \mathbf{B}_\varepsilon^n) &\leq c \left(\frac{R_k(f_0)}{r} \right)^{\frac{n}{k}} \sum_{i=0}^n \min \left(\lambda_0 \cdots \lambda_i \frac{1}{r^i} \left(\frac{r}{R_k(f_0)} \right)^{\frac{i}{k}}, \left(\frac{\varepsilon}{r} \right)^i \right) \\ &\leq c \left(\frac{R_k(f_0)}{r} \right)^{\frac{n}{k}} \left[\sum_{i=0}^{n-1} \min \left(K^i \frac{1}{r^i} \left(\frac{r}{R_k(f_0)} \right)^{\frac{i}{k}}, \left(\frac{\varepsilon}{r} \right)^i \right) + \min \left(K^{n-1} \gamma \frac{1}{r^n} \left(\frac{r}{R_k(f_0)} \right)^{\frac{n}{k}}, \left(\frac{\varepsilon}{r} \right)^n \right) \right], \end{aligned}$$

where $\lambda_0 = 1$, $c = c(n, k)$ and $R_k(f_0) = \frac{K}{(k-1)!}$. If $0 < r < 1$ and $r < \frac{R_k(f_0)\varepsilon^k}{K^k}$, then by taking the min and simplifying the right-hand side we get

$$M(r, \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) \cap \mathbf{B}_\varepsilon^n) \leq c \left(\sum_{i=0}^{n-1} K^i (R_k(f_0))^{\frac{1}{k}} \right)^{n-i} \frac{1}{r^{n-1+\frac{1}{k}}} + K^n \frac{\gamma}{r^n}.$$

When $\gamma = r^{1-\frac{1}{k}}$, we have

$$(4.6) \quad M(r, \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) \cap \mathbf{B}_\varepsilon^n) \leq c \sum_{i=0}^n K^i (R_k(f_0))^{\frac{1}{k}} \right)^{n-i} \frac{1}{r^{n-1+\frac{1}{k}}}.$$

Note that, by the definition of $M(r, A)$, it is easy to see that $M(2r, A_r) \leq M(r, A)$, where A_r denotes the r -neighborhood of subset A of \mathbb{R}^n . Therefore, if

$$(4.7) \quad M(r, \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) \cap \mathbf{B}_\varepsilon^n) m(\mathbf{B}_{2r}) < m(\mathbf{B}_\varepsilon^n),$$

where $m(A)$ denotes the Lebesgue measure of A , then there exists $v_0 \in \mathbf{B}_\varepsilon^n$, such that v_0 is not contained in a union of balls of radii $< 2r$ that covers the r -neighborhood of $\Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) \cap \mathbf{B}_\varepsilon^n$, and hence $\mathbf{B}_r^n(v_0) \cap \Delta(Df_0, \bar{\gamma}, \bar{\mathbf{B}}^n) = \emptyset$.

We want to find r , $0 < r < \min(\varepsilon, 1, \frac{R_k(f_0)\varepsilon^k}{K^k})$ satisfying 4.7. Combining 4.6 and 4.7,

we look for r satisfying

$$c \sum_{i=0}^n K^i (R_k(f_0)^{\frac{1}{k}})^{n-i} \frac{1}{r^{n-1+\frac{1}{k}}} < \frac{\varepsilon^n}{(2r)^n},$$

or

$$r < \left(\frac{\varepsilon^n}{2^n c \sum_{i=0}^n K^i (R_k(f_0)^{\frac{1}{k}})^{n-i}} \right)^{\frac{k}{k-1}}.$$

Taking

$$\begin{aligned} r(K, \varepsilon) &= \frac{1}{2} \min \left(\varepsilon, 1, \frac{R_k(f_0) \varepsilon^k}{K^k}, \left(\frac{\varepsilon^n}{2^n c \sum_{i=0}^n K^i (R_k(f_0)^{\frac{1}{k}})^{n-i}} \right)^{\frac{k}{k-1}} \right), \text{ and} \\ \gamma(K, \varepsilon) &= r(K, \varepsilon)^{1-\frac{1}{k}}, \end{aligned}$$

we get 4.7. Then we can choose $v \in \mathbf{B}_\varepsilon^n$, such that $\mathbf{B}_{\frac{r(K, \varepsilon)}{2}}^n(v) \subset \mathbf{B}_\varepsilon^n$ and every v' in $\mathbf{B}_{\frac{r(K, \varepsilon)}{2}}^n(v)$ is a $\gamma(K, \varepsilon)$ -regular value of Df_0 .

Now, let $l : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear mapping with $Dl = -v$ and $f_1 = f_0 + l$. Then $\|l\|_{C^k} \leq \varepsilon - \frac{r(K, \varepsilon)}{2}$, $Df_1 = Df_0 - v$, and $Hf_1 = Hf_0 = D(Df_0)$. So each $v' \in \mathbf{B}_{\frac{r(K, \varepsilon)}{2}}^n(v)$ is a $\gamma(K, \varepsilon)$ -regular value of Df_1 . In particular, at each critical point x_i of f_1 , we have

$$(4.8) \quad \|Hf_1(x_i)\| \geq \sigma_n(Hf_1(x_i)) \geq \gamma(K, \varepsilon).$$

In other words, the smallest absolute value of the eigenvalues of the Hessian of f_1 at its critical points is at least $\psi_1(K, \varepsilon) = \gamma(K, \varepsilon)$.

(ii) We are applying the Inverse Mapping Theorem 3.1 to $Df_1 : \overline{\mathbf{B}}^n \rightarrow \mathbb{R}^n$ at the critical points of f_1 . Let x_i be a critical point of f_1 . By (4.8) we have

$$\frac{1}{2} \frac{1}{\|Hf_1(x_i)^{-1}\|} \geq \frac{1}{2} \gamma(K, \varepsilon).$$

Choose $\delta' = \frac{1}{2} \gamma(K, \varepsilon)$, and $r' = \frac{\delta'}{K}$. Applying the Mean value theorem, when $\|x - x_i\| < r$, we have

$$\|D(Df_1)(x) - D(Df_1)(x_i)\| = \|D(Df_0)(x) - D(Df_0)(x_i)\| \leq K\|x - x_i\| < Kr' < \delta'.$$

Thus, by Theorem 3.1, Df_1 is invertible on $\mathbf{B}_{\frac{r'\delta'}{2K}}^n(x_i) = \mathbf{B}_{\frac{\gamma^2(K, \varepsilon)}{8K^2}}^n(x_i)$. Hence, $Df_1^{-1}(0) \cap \mathbf{B}_{\frac{\gamma^2(K, \varepsilon)}{8K^2}}^n(x_i)$ has only one point, i.e. x_i is the unique critical point of f_1 in the ball $\mathbf{B}_{\frac{\gamma^2(K, \varepsilon)}{8K^2}}^n(x_i)$. So the distance between any two different critical points x_i, x_j of f_1 can be estimated from below by

$$d(x_i, x_j) \geq d(K, \varepsilon) = \frac{1}{4} \frac{\gamma^2(K, \varepsilon)}{K^2}.$$

Therefore, the number of critical points of f_1 does not exceed

$$N(K, \varepsilon) = M \left(\frac{1}{4} \frac{\gamma^2(K, \varepsilon)}{K^2}, \mathbf{B}^n \right) \leq \frac{1}{8} \frac{\gamma^2(K, \varepsilon)}{K^2} n^2.$$

(iii) Suppose that the critical points of f_1 are $x_1, \dots, x_N, N \leq N(K, \varepsilon)$, and the critical values of f_1 are ordered increasingly

$$f_1(x_1) \leq f_1(x_2) \leq \dots \leq f_1(x_N).$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a C^k function satisfying the following conditions

$$g(t) = \begin{cases} 1, & \text{if } |t| < \frac{d(K, \varepsilon)}{4}, \\ 0, & \text{if } |t| > \frac{d(K, \varepsilon)}{2}, \\ 0 < g(t) < 1, & \text{if } \frac{d(K, \varepsilon)}{4} \leq |t| \leq \frac{d(K, \varepsilon)}{2}. \end{cases}$$

For each i , let $\lambda_i : \mathbb{R}^n \rightarrow [0, 1]$ be defined by $\lambda_i(x) = g(\|x - x_i\|)$, and $C_1 = \|\lambda_i\|_{C^k}$. Put $\eta_1 = \min(r(K, \varepsilon), \frac{\gamma^2(K, \varepsilon)}{8(K + \varepsilon)})$. (The second parameter of min will be used in (v)). Let

$$\lambda : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \lambda(x) = \sum_{i=1}^N c_i \lambda_i(x), \quad \text{where } c_i = (i-1) \frac{\eta_1}{4NC_1}.$$

Then $\|\lambda\|_{C^k} = \max_{1 \leq i \leq N} c_i \|\lambda_i\|_{C^k} < \frac{\eta_1}{4} \leq \frac{r(K, \varepsilon)}{4}$. Now consider the approximation of f_0 :

$$f = f_1 + \lambda = f_0 + h, \quad \text{where } h = l + \lambda.$$

We have $\|h\|_{C^k} \leq \|v\| + \|\lambda\|_{C^k} < (\varepsilon - \frac{r(K, \varepsilon)}{2}) + \frac{r(K, \varepsilon)}{4} < \varepsilon$.

Since $Df(x) = 0$ iff $Df_1(x) = -D\lambda(x)$, by the definition of λ , this equality only happens when $x \in \mathbf{B}_{\frac{d(K, \varepsilon)}{2}}^n(x_i)$ for some i . But Df_1 is injective on $\mathbf{B}_{\frac{d(K, \varepsilon)}{2}}^n(x_i)$ and $D\lambda(x) = D\lambda(x')$, when $\|x - x_i\| = \|x' - x_i\|$, so x must be equal to x_i , and then $Hf(x_i) = Hf_1(x_i) + H\lambda(x_i) = Hf_0(x_i)$. Thus f is a Morse function having the same critical points as f_1 , and $\sigma_n(Hf(x_i)) \geq \gamma(K, \varepsilon)$ for every critical point x_i .

Moreover, for any pair of distinct critical points x_i, x_j of f , we have

$$|f(x_i) - f(x_j)| = |f_1(x_i) + c_i - f_1(x_j) - c_j| \geq \psi_2(K, \varepsilon) = \frac{\eta_1(K, \varepsilon)}{4NC_1}.$$

We showed that f satisfies (i), (ii) and (iii).

(iv) Applying the Splitting Lemma 4.2 to f at each of its critical points we get

$\delta = \delta(K, \varepsilon)$ and $M(K, \varepsilon)$ satisfying (iv).

(v) First, consider $f_1 = f_0 + l$. If $\|Df_1(x)\| \leq \frac{1}{2}\eta_1 = \frac{1}{2} \min(r(K, \varepsilon), \frac{\gamma^2(K, \varepsilon)}{8(K + \varepsilon)})$, then

$\|Df_0(x) - v\| \leq \frac{r(K, \varepsilon)}{2}$, and hence, from (i) we have $\sigma_n(Hf_1(x)) = \sigma_n(Hf_0(x)) \geq \gamma(K, \varepsilon)$. According to the Inverse Mapping Theorem 3.1, Df_1 is invertible on a ball centered at x with radius $\rho_1 = \frac{(\gamma/2)^2}{2(K + \varepsilon)^2}$, and the image contains the ball centered at $Df_1(x)$ with radius $\rho_2 = \rho_1(K + \varepsilon) > \eta_1$, and hence this ball contains 0. Therefore, if $\|Df_1(x)\| \leq \frac{1}{2}\eta_1 = \frac{1}{2} \min(r(K, \varepsilon), \rho_2)$, then there exists a critical point x_i of f_1

such that $x \in \mathbf{B}_{\rho_1}^n(x_i)$.

Now consider $f = f_1 + \lambda$. If $\|Df(x)\| < \frac{1}{4}\eta_1$, then $\|Df_1(x)\| \leq \|Df(x)\| + \|D\lambda(x)\| \leq \frac{1}{4}\eta_1 + \frac{1}{4}\eta_1 = \frac{1}{2}\eta_1$, and hence $x \in \mathbf{B}_{\rho_1}^n(x_i)$. Note that $Df(x_i) = Df_1(x_i) = 0$.

Therefore, to get (v) we take

$$\eta(K, \varepsilon) = \frac{1}{4} \min(r(K, \varepsilon), \frac{\gamma^2(K, \varepsilon)}{8(K + \varepsilon)}), \text{ and } \psi_3(K, \varepsilon) = \min(\delta(K, \varepsilon), \frac{\gamma^2}{8(K + \varepsilon)^2}).$$

□

Applying the Inverse Mapping Theorem, we get a quantitative version for the openness of Morse functions on a ball as follows.

Theorem 4.5. *Let $f : \overline{\mathbf{B}}^n \rightarrow \mathbb{R}$ be a C^k function, $k \geq 2$, with $\|f\|_{C^k} \leq K$. Suppose that f is a Morse function with the critical locus $\Sigma(f) = \{x_1, \dots, x_p\}$ contained in \mathbf{B}^n and has distinct critical values. Let*

$$\begin{aligned} \gamma &= \min\{\sigma_n(Hf(x)) : x \in \Sigma(f)\}, \\ d &= \min\{|f(x_i) - f(x_j)| : i \neq j \text{ and } i, j = 1, \dots, p\}, \\ \rho &= \min(\frac{\gamma^2}{128K^2}, \frac{d}{8K}, d(\Sigma(f), \partial\mathbf{B}^n)), \text{ and} \\ \eta &= \inf\{\|Df(x)\| : x \in \overline{\mathbf{B}}^n, d(x, \Sigma(f)) \geq \rho\}. \end{aligned}$$

Let $\varepsilon = \min(\frac{\eta}{2}, \frac{\gamma^2}{64K}, \frac{d}{4})$. Then for every C^k function $\bar{f} : \overline{\mathbf{B}}^n \rightarrow \mathbb{R}$, with $\|\bar{f} - f\|_{C^k} < \varepsilon$, \bar{f} satisfies the followings:

- (i) If $x \in \mathbf{B}^n$ and $\|D\bar{f}(x)\| < \frac{\eta}{2}$, then $\sigma_n(H\bar{f}(x)) \geq \frac{\gamma}{2}$. In particular, \bar{f} is a Morse function.
- (ii) $\Sigma(\bar{f}) = \{\bar{x}_1, \dots, \bar{x}_p\} \subset \mathbf{B}^n$, and $\|\bar{x}_i - x_i\| < \rho$, for $i = 1, \dots, p$.
- (iii) $\sigma_n(H\bar{f}(\bar{x}_i)) \geq \frac{\gamma}{2}$, for $i = 1, \dots, p$.
- (iv) $\min\{|\bar{f}(\bar{x}_i) - \bar{f}(\bar{x}_j)| : i \neq j \text{ and } i, j = 1, \dots, p\} \geq \frac{d}{2}$.

Proof. First note that $\eta > 0$, since $\Sigma(f) \cap \partial\mathbf{B}^n = \emptyset$.

Let $\bar{f} : \overline{\mathbf{B}}^n \rightarrow \mathbb{R}$ be a C^k function with $\|\bar{f} - f\|_{C^k} < \varepsilon$.

Let $x \in \mathbf{B}^n$, such that $\|D\bar{f}(x)\| < \frac{\eta}{2}$. Then the definition of ε implies

$$\|Df(x)\| < \frac{\eta}{2} + \varepsilon \leq \eta,$$

and hence $d(x, \Sigma(f)) < \rho$, i.e. $\|x - x_i\| < \rho$ for some $i \in \{1, \dots, p\}$.

Moreover, by $\|\bar{f}\|_{C^k} < K + \varepsilon$ and the definition of ρ and ε , applying the Mean value theorem, we get

$$\begin{aligned} \|H\bar{f}(x) - Hf(x_i)\| &\leq \|H\bar{f}(x) - H\bar{f}(x_i)\| + \|H\bar{f}(x_i) - Hf(x_i)\| \\ &\leq (K + \varepsilon)\|x - x_i\| + \varepsilon \\ &< (K + \varepsilon)\rho + \varepsilon \\ &\leq (1 + \rho)\varepsilon + K\rho < \frac{\gamma}{2}. \end{aligned}$$

Therefore $\sigma_n(H\bar{f}(x)) \geq \frac{\gamma}{2}$. This proves (i) and (iii).

For $x_i \in \Sigma(f)$, we have $\|D\bar{f}(x_i)\| < \varepsilon \leq \frac{\eta}{2}$. From (i) and the Inverse Mapping Theorem 3.1, $D\bar{f}$ is invertible on a ball centered at x_i with radius $\frac{(\gamma/4)^2}{2(K+\varepsilon)^2} \geq \frac{\gamma^2}{128K^2} \geq$

ρ , and the image contains a ball centered at $D\bar{f}(x_i)$ with radius $\frac{(\gamma/4)^2}{2(K+\varepsilon)} > \frac{\gamma^2}{64K} \geq \varepsilon > \|D\bar{f}(x_i)\|$. From these facts and (i), there exists $\bar{x} \in \Sigma(\bar{f})$ such that $\|\bar{x} - x_i\| < \rho$. Note that, by the definition of ρ , all critical points of \bar{f} are contained in \mathbf{B}^n .

Moreover, for any two distinct critical points $\bar{x}, \bar{y} \in \Sigma(\bar{f})$, $\|\bar{x} - \bar{y}\| > 2\rho$. Hence for all $x_i \in \Sigma(f)$ there exists only one $\bar{x}_i \in \Sigma(\bar{f})$ such that $\|\bar{x}_i - x_i\| < \rho$, (ii) follows.

To prove (iv), adding a constant to \bar{f} , we can assume $\bar{f}(0) = f(0)$. Then, by the Mean value theorem, $|\bar{f}(x) - f(x)| < \varepsilon$, for all $x \in \mathbf{B}^n$. So for $i = 1, \dots, p$, we have

$$\begin{aligned} |\bar{f}(\bar{x}_i) - f(x_i)| &\leq |\bar{f}(\bar{x}_i) - f(\bar{x}_i)| + |f(\bar{x}_i) - f(x_i)| \\ &< \varepsilon + K\|\bar{x}_i - x_i\| \leq \frac{d}{4} + K\rho < \frac{d}{2}. \end{aligned}$$

By the triangle inequality, for any two distinct critical points $\bar{x}_i, \bar{x}_j \in \Sigma(\bar{f})$, we have

$$|\bar{f}(\bar{x}_i) - \bar{f}(\bar{x}_j)| \geq \frac{d}{2}.$$

□

Acknowledgement. This research is supported by Vietnamese National Foundation for Science and Technology Development 2010 - 2012.

REFERENCES

- [A-G-V] V. I. Arnold, M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps, Vol. I, Monograph Math., 82, Birkhäuser (1985).
- [A-M-R] R. Abraham, J. E. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications, Springer (2001).
- [B-L] Th. Bröcker and L. Lander, Differentiable Germs and Catastrophes, LMS Lecture Notes Series, 17, Cambridge University Press (1976).
- [C1] F. H. Clarke, *On the inverse function theorem*, Pacific Journal of Mathematics, Vol 64, No 1 (1976), 97-102.
- [C2] F. H. Clarke, Optimization and Nonsmooth Analysis, Classics in Applied Mathematics, Vol. 5, Wiley (1983).
- [F] H. Federer, Geometric measures theory, Springer-Verlag (1969).
- [G-G] M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities, Texts in Math. 14, AMS (1973).
- [G-L] G. H. Golub and C. F. van Loan, Matrix computation, Johns Hopkins Univ. Press (1983).
- [L-P] T. L. Loi and P. Phien, *The quantitative Morse theorem*, Int. Journal of Math. Analysis, Vol. 6, No. 10 (2012), 481-491.
- [Ma] J. Martinet, Singularities of Smooth Functions and Maps, Cambridge Univ. Press (1982).
- [Pa] M. Papi, *On the domain of the Implicit Function and applications*, Journal of Inequalities and its Applications, 3 (2005), 221-234.
- [P] P. Phien, *Some quantitative results on Lipschitz inverse and implicit function theorems*, East-West Journal of Mathematics, Vol. 13, No 1 (2011), 7-22.

- [Y1] Y. Yomdin, *The Geometry of Critical and Near-Critical Values of Differentiable Mappings*, Math. Ann. 264, (1983), 495-515.
- [Y2] Y. Yomdin, *Some quantitative results in singularity theory*, Annales Polonici Mathematici, 37 (2005), 277-299.
- [Y-C] Y. Yomdin and G. Comte, *Tame geometry with application in smooth analysis*, LNM vol. 1834 (2004).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DALAT, DALAT, VIETNAM
E-mail address: loitl@dlu.edu.vn

NHATRANG COLLEGE OF EDUCATION, 1 NGUYEN CHANH, NHATRANG, VIETNAM
E-mail address: phieens@yahoo.com